

Stability of bounded global solutions for Navier-Stokes equations ^{*†}

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Abstract

In this paper some kind of asymptotic behavior of the solutions for the Navier-Stokes system on \mathbb{R}^n in abstract Banach spaces is studied under the existence of global in time solutions. The asymptotic stability of the zero solution is also shown.

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1 Introduction

The Navier -Stokes equations describing the motion of an incompressible fluid in \mathbb{R}^n , $n \geq 2$, without external forces are written as follows

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla)u + \nabla p &= 0 \\ \nabla \cdot u &= 0, \\ u(0) &= u_0. \end{aligned} \tag{1}$$

Here $u(x, t) = (u_1(x, t), \dots, u_n(x, t))$ is the unknown velocity of the fluid and $p = p(x, t)$ is the unknown pressure at the point $x \in \mathbb{R}^n$ and time $t \geq 0$.

Let $u_0(x)$ be an initial condition that verifies $\nabla \cdot u_0 = 0$.

As usual, \mathbf{P} denotes the projection from $(L^2(\mathbb{R}^n))^n$ onto the subspace $\mathbf{P}(L^2(\mathbb{R}^n))^n = \{f \in L^2 : \nabla \cdot f = 0\}$ of solenoidal vector fields

$$\mathbf{P}(u_1, \dots, u_n) = (u_1 - R_1 \sigma, \dots, u_n - R_n \sigma),$$

where R_j are the Riesz transforms, with symbols $\xi_j/|\xi_j|$ and $\sigma = R_1 u_1 + \dots + R_n u_n$.

It is well known that \mathbf{P} can be extended to a continuous operator on $L^p(\mathbb{R}^n)$ to $\mathbf{P}(L^p(\mathbb{R}^n)) = \{f \in L^p(\mathbb{R}^n) : \nabla \cdot f = 0\}$. (c fr [Fuyiwara, Morimoto(1977)])

Using \mathbf{P} the equations (1) can be rewritten as

$$\begin{aligned} u_t &= \Delta u - \mathbf{P} \nabla(u \otimes u) \\ \nabla \cdot u &= 0 \\ u(0) &= u_0. \end{aligned} \tag{2}$$

$(u \cdot \nabla)u$ can be replaced by $\nabla(u \otimes u)$, because $\nabla \cdot u = 0$.

The heat semigroup $S(t)$ is given by the convolution with the Weierstrass kernel, or heat kernel, $S(t) = e^{t\Delta} = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} *$; then the problem (2) can be written under the following integral form

$$u(t) = S(t)u_0 - \int_0^t \mathbf{P} \nabla S(t - \tau)(u \otimes u)(\tau) d\tau \quad (3)$$

where the integrals are understood in the Bochner's sense. The solutions of this integral equation are called mild solutions.

Then, a solution of (1) or (2) will be interpreted as an E valued mapping defined in $[0, \infty)$, for an appropriate Banach space E .

In this article, by solutions of the Navier-Stokes system we mean solutions of type (3).

The goal of this paper is to show a stability result for global in time solutions for (1). In theorem 2 the difference of two given bounded solutions in an abstract Banach space goes to zero with the precise rate of decay. As an immediate consequence the asymptotic stability of the zero solution is obtained.

In section two we present the mathematical setting to build the solutions mentioned above. Section three is dedicated to state and prove our main theorem.

2 Abstract Banach spaces

In this section we retrieve all the needed definitions to construct a Banach space adequate to the Navier-Stokes system. These ideas were introduced in [Cannone, Karch(2001)], [Karch(1999)] and even [Lemarié-Rieusset(2002)].

Definition 2.1. *A Banach space $(E, \|\cdot\|_E)$ is said to be functional and translation invariant if the following three conditions are satisfied:*

- (i) $S \subset E \subset S'$ and both inclusions are continuous,
- (ii) for every $f \in E$, $\tau : \mathbb{R}^n \rightarrow E$ defined by $\tau_y f(x) = f(x + y)$ is measurable in the sense of Bochner with respect to the Lebesgue measure on \mathbb{R}^n ,
- (iii) the norm $\|\cdot\|_E$ on E is invariant translation

$$\forall f \in E, y \in \mathbb{R}^n \quad \|\tau_y f\|_E = \|f\|_E.$$

Definition 2.2. *We call the space $(E, \|\cdot\|_E)$ adequate to the problem (1) if*

- (i) $(E, \|\cdot\|_E)$ is a functional and translation invariant Banach space.

(ii) $\forall f, g \in E$ the product $f \otimes g$ is well defined as a tempered distribution. Moreover, there exists $T_0 > 0$ and a positive function $w \in L^1(0, T_0)$ such that

$$\|\mathbf{P}\nabla S(\tau)(f \otimes g)\|_E \leq w(\tau)\|f\|_E\|g\|_E, \quad \forall f, g \in E, \tau \in (0, T_0).$$

Some examples of Banach spaces adequate to (1) are the subspaces of free divergence functions of the spaces L^p , L_w^p , $L^{p,q}$, M_q^p with $p > n$ and $q \geq 1$.

In this paper, the norms of the Banach spaces have additional scaling properties. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be, the rescaled function f_λ is defined by

$$f_\lambda(x) = f(\lambda x), \quad \lambda > 0.$$

This definition is extended for all $f \in S'$ in the usual way.

Definition 2.3. Let $(E, \|\cdot\|_E)$ be a Banach space which can be imbedded continuously in S' . The norm $\|\cdot\|_E$ is said to have a scaling degree equal to k if

$$\|f_\lambda\|_E = \lambda^k \|f\|_E, \quad \forall f \in E \text{ such that } f_\lambda \in E \text{ and } \forall \lambda > 0.$$

Remark: The usual norms of the spaces L^p , L_w^p , $L^{p,q}$, M_q^p have scaling degree equal to $-n/p$. Moreover, the standard norm in the homogeneous Sobolev space $\dot{H}^s = \{f \in S' : |\xi|^s \hat{f}(\xi) \in L^2\}$ has scaling degree equal to $s - n/2$.

A Banach space E endowed with a norm with scaling degree equal to $-n/p$, $p > n$ will be denoted by E_p .

Let $E \subset S'$ be a Banach space, in [Karch(1999)] was introduced the space of distributions BE^β which is an “homogeneous Besov space modelled on E ”.

Definition 2.4. Let $\beta \geq 0$. Given a Banach space E continuously imbedded in S' , define

$$BE^\beta = \{f \in S' : \|f\|_{BE^\beta} = \sup_{t>0} t^{\beta/2} \|S(t)f\|_E < \infty\}.$$

For instance, for the Banach space $E = L^p$, the norm $\|\cdot\|_{BE^\beta}$ is equivalent to the standard norm of the homogeneous Besov spaces $\dot{B}_{p,\infty}^{-\beta}$.

It is straightforward to see that if E has a norm with scaling degree equal to k , then $\|\cdot\|_{BE^\beta}$ has scaling degree equal to $k - \beta$. Indeed, given any $f \in S'$ and $\lambda > 0$,

$$S(t)f_\lambda = (S(\lambda t^2)f)_\lambda$$

then,

$$\|f_\lambda\|_{BE^\beta} = \sup_{t>0} t^{\beta/2} \|S(t)f_\lambda\|_E = \lambda^{k-\beta} \sup_{\lambda^2 t > 0} (\lambda^2 t)^{\beta/2} \|S(\lambda^2 t)f\|_E = \lambda^{k-\beta} \|f\|_{BE^\beta}.$$

It is well known that for the case of a Banach space E_p such that for some $q \in [1, \infty]$, $e^{t\Delta} : E_p \rightarrow L^q$ is a bounded operator for every $t > 0$, then $(BE_p^\beta, \|\cdot\|_{BE_p^\beta})$ is a Banach space.

These definitions allow us to construct global in time solutions (for small initials conditions) in the space $\chi \equiv \mathcal{C}([0, \infty), BE_p^\beta)$ consisting of measurable and essentially bounded functions $u : [0, \infty) \rightarrow BE_p^\beta$ such that $u(t) \rightarrow u(0)$ as $t \searrow 0$ in the topology of S' .

3 Global in time solutions to the Navier-Stokes equations

In this section the main result is exposed. For this purpose, we recall the following two statements. In the first lemma the boundedness property of the operator $\mathbf{P}\nabla S(t)(u \otimes v) = e^{t\Delta} B(u, v)$, where $B(u, v) = \mathbf{P}\nabla(u \otimes v)$, is shown.

Theorem 1 gives the conditions for the existence and unicity of global solutions. Although this result is independent of our main statement (Theorem 2), we decided to include it here just to keep in mind that the set of global solutions of system (1) is neither empty nor single (with zero as the unique global solution).

Lemma 3.1. *[Karch(1999)] Assume that the Banach space E_p is adequate to the problem (1) and has a norm with scaling degree equal to $-n/p$. Then*

1. *There exists a constant $C_1 > 0$ independent of t, u, v , such that*

$$\|e^{t\Delta}B(u, v)\|_{E_p} \leq C_1 t^{-(1+n/p)/2} \|u\|_{E_p} \|v\|_{E_p}$$

for all $u, v \in E_p$ and $t > 0$.

2. Let $0 \leq \beta \leq 1 + n/p$. There exists a constant $C_2 > 0$, independent of t, u, v such that

$$\|e^{t\Delta}B(u, v)\|_{BE_p^\beta} \leq C_2 t^{(\beta-1-n/p)/2} \|u\|_{E_p} \|v\|_{E_p}$$

In the following theorem global in time solutions are built for the problem (1) in the space BE_p^β considering small initial conditions. The proof of this result is obtained by handling standard tools introduced in [Kato(1984)] and used by other authors. See, for instance, [Barraza(1999)], [Cannone(1995)], [Cannone, Karch(2001)], or [Lemarié-Rieusset(2002)].

Theorem 3.2. [Karch(1999)] Fix $p > n, n > 0$. Denote $\beta = 1 - n/p$. Let E_p be a Banach space satisfying

- (i) E_p is an adequate space to the problem (1);
- (ii) the norm $\|\cdot\|_{E_p}$ has scaling degree equal to $-n/p$;
- (iii) there exists $q > 0$ such that the operator $e^{t\Delta} : E_p \rightarrow L^q$ is bounded for every $t > 0$.

Then, there exists $\epsilon > 0$ such that for every $u_0 \in BE_p^\beta$ satisfying $\|u_0\|_{BE_p^\beta} < \epsilon$ there exists a solution of (1) for all $t > 0$ in the space

$$\chi \equiv \mathcal{C}([0, \infty), BE_p^\beta) \cap \{u : (0, \infty) \rightarrow E_p : \sup_{t>0} t^{\beta/2} \|u(t)\|_{E_p} < \infty\}.$$

This is the unique solution satisfying the condition $\sup_{t>0} t^{\beta/2} \|u(t)\|_{E_p} \leq 2\epsilon$.

Now, we are able to state our main result.

Theorem 3.3. Fix $p > n, n > 0$. Denote $\beta = 1 - n/p$. Let E_p be a Banach space satisfying the same hypothesis (i), (ii), (iii) as in the previous theorem. Let u, v be two global in time solutions of (1) in the space χ corresponding to the initial conditions $u_0, v_0 \in BE_p^\beta$ respectively. Suppose that $\|u\|_\chi \leq M$ and $\|v\|_\chi \leq M$, for some constant $M > 0$.

Let $w_0 = u_0 - v_0$ and $w(t) = u(t) - v(t)$.

Then

$$\|w(t)\|_{E_p} \leq (\|w_0\|_{BE_p^\beta} + 4CM^2)t^{-\beta/2}, \quad t > 0.$$

That is, $\|u(t) - v(t)\|_{E_p}$ goes asymptotically to 0 like $t^{-\beta/2}$.

Remark: This theorem plays a fundamental role in providing a way to compare two bounded global in time solutions for the system (1), corresponding to any initial conditions, even if the norms of these initial conditions are not small enough.

Proof

Let the space

$$\chi \equiv \mathcal{C}([0, \infty), BE_p^\beta) \cap \{u : (0, \infty) \rightarrow E_p : \sup_{t>0} t^{\beta/2} \|u(t)\|_{E_p} < \infty\}$$

be endowed with the norm

$$\|u\|_\chi = \max\left\{\sup_{t>0} \|u(t)\|_{BE_p^\beta}, \sup_{t>0} t^{\beta/2} \|u(t)\|_{E_p}\right\}$$

Let u, v two global in time solutions of (1) corresponding to the initial data $u_0, v_0 \in BE_p^\beta$, respectively.

We know that

$$\begin{aligned} u(t) &= S(t)u_0 - \int_0^t \mathbf{P} \nabla S(t-\tau)(u \otimes u)(\tau) d\tau \\ &= e^{t\Delta} u_0 - \int_0^t e^{(t-\tau)\Delta} B(u, u) d\tau. \end{aligned}$$

Analogously,

$$v(t) = e^{t\Delta} v_0 - \int_0^t e^{(t-\tau)\Delta} B(v, v) d\tau$$

and hence,

$$u(t) - v(t) = e^{t\Delta}(u_0 - v_0) - \int_0^t e^{(t-\tau)\Delta} [B(u, u) - B(v, v)] d\tau.$$

From this, we have

$$\begin{aligned}
t^{\beta/2} \|u(t) - v(t)\|_{E_p} &= t^{\beta/2} \|w(t)\|_{E_p} \\
&\leq t^{\beta/2} \|e^{t\Delta}(u_0 - v_0)\|_{E_p} \\
&\quad + t^{\beta/2} \int_0^t \|e^{(t-\tau)\Delta}[B(u, u-v) + B(u-v, v)]\|_{E_p} d\tau \\
&\leq \|w_0\|_{BE_p^\beta} \\
&\quad + t^{\beta/2} \int_0^t [\|e^{(t-\tau)\Delta}B(u, u-v)\|_{E_p} + \|e^{(t-\tau)\Delta}B(u-v, v)\|_{E_p}] d\tau.
\end{aligned}$$

Applying lemma 1 we can conclude

$$\begin{aligned}
t^{\beta/2} \|u(t) - v(t)\|_{E_p} &\leq \|w_0\|_{BE_p^\beta} \\
&\quad + t^{\beta/2} \int_0^t C(t-\tau)^{-(1+n/p)/2} [\|u(\tau)\|_{E_p} \|(u-v)(\tau)\|_{E_p} \\
&\quad + \|v(\tau)\|_{E_p} \|(u-v)(\tau)\|_{E_p}] d\tau \\
&\leq \|w_0\|_{BE_p^\beta} \\
&\quad + C(\|u\|_\chi + \|v\|_\chi) t^{\beta/2} \int_0^t (t-\tau)^{-(1+n/p)/2} \tau^{-\beta} (\tau^{\beta/2} \|(u-v)(\tau)\|_{E_p}) d\tau \\
&\leq \|w_0\|_{BE_p^\beta} + C(\|u\|_\chi + \|v\|_\chi) (\sup_{\tau>0} \tau^{\beta/2} \|(u-v)(\tau)\|_{E_p}) \\
&\leq \|w_0\|_{BE_p^\beta} + C(2M)^2.
\end{aligned}$$

Therefore,

$$\|w(t)\|_{E_p} = \|u(t) - v(t)\|_{E_p} \leq (\|w_0\|_{BE_p^\beta} + 4CM^2) t^{-\beta/2}.$$

Corollary 3.4. *Under the hypotheses of Theorem 2 the zero solution is asymptotically*

stable. More precisely, let u be a bounded global solution of (1) with initial condition u_0 , the norm $\|u\|_{E_p}$ tends to zero as $t^{-\beta/2}$.

Proof

It is immediately to show this result after taking $v = 0$ in the previous theorem.

Remark: This property of the zero solution holds in almost all the known solutions spaces for the Navier-Stokes equations.

Corollary 3.5. *Let the assumptions from the Theorem 2 hold. Let u, v be two solutions of (1) corresponding to the initial data $u_0, v_0 \in BE_p^\beta$ respectively. Suppose that*

$$\lim_{t \rightarrow \infty} t^{\beta/2} \|e^{t\Delta}(u_0 - v_0)\|_{E_p} = 0.$$

Then,

$$\lim_{t \rightarrow \infty} t^{\beta/2} \|u(\cdot, t) - v(\cdot, t)\|_{E_p} = 0$$

provided that $M > 0$ is sufficiently small.

Before proving this statement, let us recall the following technical lemma.

Lemma 3.6. *[Barraza(1999)] Let $w \in L^1(0, 1)$ and $\int_0^1 w(x)dx < 1$. Assume that f and g are two nonnegative, bounded measurable functions such that*

$$f(t) \leq g(t) + \int_0^1 w(\tau)f(\tau t)d\tau.$$

Then $\lim_{t \rightarrow \infty} g(t) = 0$ implies $\lim_{t \rightarrow \infty} f(t) = 0$.

Proof of Corollary 2

As in Theorem 2,

$$t^{\beta/2} \|u(t) - v(t)\|_{E_p} \leq t^{\beta/2} \|e^{t\Delta}(u_0 - v_0)\|_{E_p}$$

$$+ t^{\beta/2} \int_0^t C(t - \tau)^{-(1+n/p)/2} [\|u(\tau)\|_{E_p} \|(u - v)(\tau)\|_{E_p}$$

$$+ \|v(\tau)\|_{E_p} \|(u - v)(\tau)\|_{E_p}] d\tau.$$

Since the solutions u, v are bounded with bound M , we have

$$\begin{aligned} t^{\beta/2} \|u(t) - v(t)\|_{E_p} &\leq t^{\beta/2} \|e^{t\Delta}(u_0 - v_0)\|_{E_p} \\ &\quad + 2MCt^{\beta/2} \int_0^t (t - \tau)^{-(1+n/p)/2} \tau^{-\beta} (\tau^{\beta/2} \|(u - v)(\tau)\|_{E_p}) d\tau. \end{aligned}$$

After a change of variables, it is possible to write

$$\begin{aligned} t^{\beta/2} \|u(t) - v(t)\|_{E_p} &\leq t^{\beta/2} \|e^{t\Delta}(u_0 - v_0)\|_{E_p} \\ &\quad + 2CM \int_0^1 (1 - s)^{-(1+n/p)/2} s^{-\beta} (ts)^{\beta/2} \|(u - v)(ts)\|_{E_p} ds. \end{aligned}$$

Putting $f(t) = t^{\beta/2} \|u(t) - v(t)\|_{E_p}$ and using that $(1 - s)^{-(1+n/p)/2} s^{-\beta} \in L^1(0, 1)$ we may apply Lemma 3 to obtain $t^{\beta/2} \|u(t) - v(t)\|_{E_p} \rightarrow 0$ as $t \rightarrow \infty$ for sufficiently small $M > 0$.

References

- [Barraza(1999)] Barraza, O., Regularity and stability for the solutions of the Navier-Stokes equations in Lorentz spaces, *Nonlinear Analysis* 35, 1999, 747-764, 1999.
- [Cannone(1995)] Cannone, M., *Ondelettes, paraproducts et Navier-Stokes*, Diderot, Paris, 1995.
- [Cannone, Karch(2001)] Cannone, M.; Karch, G., Incompressible Navier-Stokes equations in abstract Banach spaces, in “Tosio Kato’s Method and Principle for Evolution Equations in Mathematical Physics”, Edited by H. Fujita, S. T. Kuroda, and H. Okamoto, 2001.
- [Fuyiwara, Morimoto(1977)] Fuyiwara, D.; Morimoto, A., An L_r -theorem of the Helmholtz decomposition of vector fields, *J. Fac. Univ. Tokyo, Sec. I* 24, 1977, 685-700.
- [Karch(1999)] Karch, G., Scaling in nonlinear parabolic equations, *Journal of Mathematical Analysis and Applications* 234, 1999, 534-558.

[Kato(1984)] Kato, T., Strong L^p solutions of the Navier Stokes equation in \mathbb{R}^n , with applications to weak solutions, Math. Z. 187,1984, 471-480.

[Lemarié-Rieusset(2002)] Lemarié-Rieusset, P. G., Recent developments in the Navier-Stokes problem, Chapman & Hall/ CRC Press, 2002.